# On the Construction of Zero Energy States in Supersymmetric Matrix Models IV*

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**Abstract**

Simple recursion relations for zero energy states of supersymmetric matrix models are derived by using an unconventional *reducible* representation for the fermionic degrees of freedom.

## 1 The model

The supercharges of the model under consideration,

$$Q_\beta = \left(-i\partial_{tA}\gamma^t_{\beta\alpha} + \tfrac{1}{2}f_{ABC}q_{sB}q_{tC}\gamma^{st}_{\beta\alpha}\right)\theta_{\alpha A}, \tag{1}$$

with

$$\{\theta_{\alpha A}, \theta_{\beta B}\} = 2\delta_{\alpha\beta}\delta_{AB}, \tag{2}$$

satisfy the supersymmetry algebra

$$\{Q_\beta, Q_{\beta'}\} = \delta_{\beta\beta'}2H + 4\gamma^t_{\beta\beta'}q_{tA}J_A, \tag{3}$$

where

$$\begin{aligned} H &= -\Delta + V + \tfrac{1}{2}W_{\alpha A,\beta B}\theta_{\alpha A}\theta_{\beta B}, \\ V &= -\tfrac{1}{2}\textstyle\sum_{s,t=1}^{d} \operatorname{tr}\,[X_s, X_t]^2, \\ W_{\alpha A,\beta B} &= i f_{ABC}q_{tC}\gamma^t_{\alpha\beta} \end{aligned} \tag{4}$$

and

$$J_A = -if_{ABC}\left(q_{sB}\partial_{sC} + \tfrac{1}{4}\theta_{\alpha B}\theta_{\alpha C}\right) = L_A + S_A. \tag{5}$$

As each of the $Q_\beta$ squares to $H$ on gauge-invariant states $\Psi$, i.e. when $J_A\Psi = 0$ ($A = 1, \dots, N^2-1$ in the case of SU($N$)), it is convenient to sometimes suppress the index $\beta$ (which, corresponding to the dimensions of the real representations for the $\gamma$'s, takes $s_d = 2(d-1)$ different values iff $d = 2$, 3, 5 or 9).

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Writing

$$Q_\beta =: D_{\alpha A}\theta_{\alpha A} = \sum_{a=1}^{2\Lambda} D_a\theta_a, \tag{6}$$

with $2\Lambda := s_d(N^2-1)$ (the total number of fermionic degrees of freedom), and choosing $\gamma^d$ to be diagonal, it immediately follows from (3) and

$$\{Q_\beta, Q_{\beta'}\} = \{D_a\theta_a, D'_b\theta_b\} = [D_a, D'_b]\theta_a\theta_b + D'_b D_a\{\theta_a,\theta_b\} \tag{7}$$

that the differential operators $D_a$ ($\beta=\beta'$) satisfy

$$D_aD_a = -\Delta + V \pm 2q_{dC}L_C \tag{8}$$

$$[D_a, D_b] = W_{ab} \pm 4q_{dC}S_C(a,b), \quad S_C(\alpha A, \beta B) = -\tfrac{i}{4}f_{ABC}\delta_{\alpha\beta} \tag{9}$$

and the $\pm$ sign corresponds to

$$\gamma^d_{\beta\beta} = \begin{cases} +1 \text{ for } \beta \le \frac{s_d}{2}, \text{ say, and} \\ -1 \text{ for } \beta > \frac{s_d}{2}. \end{cases}$$

## 2 Recursive solution in the left-action representation

Consider the (reducible) representation of (2) in which the $\theta$s act by multiplication from the left on the Clifford algebra they generate, i.e. on the vector space $\mathscr{P}$ of polynomials

$$\begin{aligned}\Psi &= \psi + \psi_a\theta_a + \tfrac{1}{2}\psi_{ab}\theta_a\theta_b + \ldots + \tfrac{1}{(2\Lambda)!}\psi_{a_1\ldots a_{2\Lambda}}\theta_{a_1}\ldots\theta_{a_{2\Lambda}} \\ &= \textstyle\sum_{k=0}^{2\Lambda}\frac{1}{k!}\psi_{a_1\ldots a_k}\theta_{a_1}\theta_{a_2}\ldots\theta_{a_k},\end{aligned} \tag{10}$$

where the coefficients $\psi_{a_1\ldots a_k}$ are totally antisymmetric in their indices. The (graded) Hilbert space of the model, $\mathscr{H} = \oplus_{k=0}^{2\Lambda}\mathscr{H}_k = \mathscr{H}_+ \oplus \mathscr{H}_-$, is spanned by such polynomials with $\psi_{a_1\ldots a_k} \in L^2(\mathbb{R}^{d(N^2-1)})$, so that $\Psi$ normalizable corresponds to[1]

$$\int |\psi_{a_1\ldots a_k}(q)|^2 \prod_{t,A} \mathrm{d}q_{tA} < \infty \quad \forall k. \tag{11}$$

The dimension of this representation ($\dim\mathscr{P} = 2^{2\Lambda}$) is vastly greater than that of the irreducible one, but it is completely reducible – breaking up block-diagonally into the direct sum of $2^\Lambda$ copies of the irreducible one. Hence, any non-trivial solution of $H\Psi = 0$ in $\mathscr{H}$ would imply the existence of a zero-energy state in the Hilbert space $\hat{\mathscr{H}}$ corresponding to the conventional irreducible representation.

Letting $Q_\beta$ act on $\mathscr{H}_+$ (the even-grade part of $\mathscr{H}$), $Q_\beta\Psi = 0$ amounts to[2]

$$D_{[a}\psi_{a_1\ldots a_{2k}]} = \tfrac{1}{2k+1}D_c\psi_{aca_1\ldots a_{2k}}, \tag{12}$$

[1]One can define the scalar product in $\mathscr{H}$ e.g. by $\langle\Phi,\Psi\rangle = \int\langle\Phi^*_{\mathrm{rev}}\Psi\rangle_0$, where $(\cdot)_{\mathrm{rev}}$ denotes reversion of the order of $\theta$s, $(\cdot)^*$ complex conjugation, and $\langle\cdot\rangle_0$ projection onto grade 0 in $\mathscr{P}$.

[2]Cp. [1] for the corresponding irreducible (but manifest SO($d$)-invariance breaking) formulation.

i.e.

$$D_a\psi_{a_1\ldots a_{2k}} + D_{a_1}\psi_{a_2\ldots a_{2k}a} + \ldots + D_{a_{2k}}\psi_{aa_1\ldots a_{2k-1}} = D_c\psi_{aca_1\ldots a_{2k}}. \tag{13}$$

Acting on (13) with $D_a$ and summing over $a$ gives $(-\Delta + V \pm 2q_{dC}L_C)\psi_{a_1\ldots a_{2k}}$ for the first term, and $\frac{1}{2}(W_{ac} \pm 4q_{dC}S_C(a,c))\psi_{aca_1\ldots a_{2k}}$ on the right hand side. What about the $2k$ remaining terms $2kD_aD_{[a_1}\psi_{a_2\ldots a_{2k}]a}$? One has

$$\begin{aligned} D_aD_{a_1}\psi_{a_2\ldots a_{2k}a} &= D_{a_1}D_a\psi_{a_2\ldots a_{2k}a} + (W_{aa_1} \pm \ldots)\psi_{a_2\ldots a_{2k}a} \\ &= (2k-1)D_{a_1}D_{[a_2}\psi_{a_3\ldots a_{2k}]} + (W_{aa_1} \pm \ldots)\psi_{a_2\ldots a_{2k}}, \end{aligned} \tag{14}$$

using $(12)_{k\to k-1}$; so

$$2kD_aD_{[a_1}\psi_{a_2\ldots a_{2k}]a} = 2kW_{a[a_1}\psi_{a_2\ldots a_{2k}]a} \pm \ldots + (2k-1)(2k)D_{[a_1}D_{a_2}\psi_{a_3\ldots a_{2k}]}, \tag{15}$$

where the last antisymmetrized expression again equals $\frac{1}{2}(W_{[a_1a_2} \pm \ldots)\psi_{a_3\ldots a_{2k}]}$. The terms containing the bosonic $L_A$ and fermionic $S_A$ can either be shown to cancel using the assumption $J_A\Psi = 0$, or one simply adds the equations resulting for $\beta \leq s_d/2$ to the ones resulting for $\beta > s_d/2$. In any case, what one can also obtain this way are of course the equations that result by considering $H\Psi = 0$ directly:

$$\begin{aligned} &(-\Delta + V)\psi_{a_1\ldots a_{2k}} + 2kW_{a[a_1}\psi_{a_2\ldots a_{2k}]a} + k(2k-1)W_{[a_1a_2}\psi_{a_3\ldots a_{2k}]} \\ &\quad = \tfrac{1}{2}W_{ac}\psi_{aca_1\ldots a_{2k}}. \end{aligned} \tag{16}$$

Their recursive solution could proceed as follows: The lowest-grade equation $(-\Delta + V)\psi = \frac{1}{2}W_{ac}\psi_{ac}$ yields

$$\psi = \tfrac{1}{2}(-\Delta + V)^{-1}W_{ac}\psi_{ac}. \tag{17}$$

Using (17) to replace $\psi$ in $(16)_{k=1}$, …, respectively $\psi_{a_3\ldots a_{2k}}$ in $(16)_k$ via the analogue of (17),

$$\psi_{a_3\ldots a_{2k}} = \tfrac{1}{2}(H_{2k-2}^{-1}W_{ac}\psi_{ac})_{a_3\ldots a_{2k}}, \tag{18}$$

(16) takes the form

$$(H_{2k}\Psi)_{a_1\ldots a_{2k}} = \tfrac{1}{2}W_{ac}\psi_{aca_1\ldots a_{2k}}, \tag{19}$$

with $H_{2k}$ only acting on $\Psi_{2k} \in \mathscr{H}_{2k}$. This procedure is based on the fact that $H_0 = -\Delta + V$ is invertible and the assumption that this also holds for higher-grade $H_{2k}$ on $\mathscr{H}_{2k}$.

# 3 Recursion relations in a diagonalizing basis

Note that

$$\tfrac{1}{2}W_{ab}\theta_a\theta_b\left(\psi + \tfrac{1}{2}\psi_{a_1a_2}\theta_{a_1}\theta_{a_2} + \tfrac{1}{4!}\psi_{a_1a_2a_3a_4}\theta_{a_1}\theta_{a_2}\theta_{a_3}\theta_{a_4} + \ldots\right) \stackrel{!}{=} \mu(q)\Psi \tag{20}$$

gives the set of equations

$$\begin{aligned} &\tfrac{1}{2}W_{a_2a_1}\psi_{a_1a_2} = \mu\psi \\ &W_{a_1a_2}\psi + W_{a_1a}\psi_{aa_2} - W_{a_2a}\psi_{aa_1} + \tfrac{1}{2}W_{ab}\psi_{baa_1a_2} = \mu\psi_{a_1a_2} \\ &\vdots \end{aligned} \tag{21}$$

while $H\Psi \stackrel{!}{=} 0$ in the left-action representation gives

$$
\begin{aligned}
&(-\Delta + V)\psi = \tfrac{1}{2} W_{ac}\psi_{ac} \\
&(-\Delta + V)\psi_{a_1 a_2} + W_{aa_1}\psi_{a_2 a} - W_{aa_2}\psi_{a_1 a} \\
&\quad + W_{a_1 a_2}\tfrac{1}{2}(-\Delta + V)^{-1} W_{ab}\psi_{ab} = \tfrac{1}{2} W_{ac}\psi_{aca_1 a_2} \\
&\vdots
\end{aligned}
\tag{22}
$$

These equations can be simplified by performing a (pointwise) diagonalization $W = UDU^{-1}$, where

$$
\begin{gathered}
U = [w_1, w_2, \dots, w_\Lambda, w_1^*, \dots, w_\Lambda^*], \\
D = \mathrm{diag}(\lambda_1, \dots, \lambda_{2\Lambda}) = \mathrm{diag}(\mu_1, \dots, \mu_\Lambda, -\mu_1, \dots, -\mu_\Lambda), \quad (\mu_k \geq 0).
\end{gathered}
\tag{23}
$$

Corresponding to changing to the space-dependent (non-hermitian) fermion basis

$$\tilde{\theta}_a := (U^\dagger)_{ac}\theta_c = U^*_{ca}\theta_c \tag{24}$$

which diagonalizes the fermionic part of the hamiltonian,

$$H_F = \tfrac{1}{2} W_{ab}\theta_a\theta_b = \tfrac{1}{2}\textstyle\sum_c \lambda_c \tilde{\theta}_c^\dagger \tilde{\theta}_c, \tag{25}$$

one could introduce

$$\tilde{\psi}_{\tilde{a}_1 \dots \tilde{a}_n} := (U^T)_{\tilde{a}_1 a_1} \dots (U^T)_{\tilde{a}_n a_n} \psi_{a_1 \dots a_n}, \tag{26}$$

i.e. substitute

$$\psi_{a_1 \dots a_n} = (U^*)_{a_1 \tilde{a}_1} \dots (U^*)_{a_n \tilde{a}_n} \tilde{\psi}_{\tilde{a}_1 \dots \tilde{a}_n} \tag{27}$$

in all equations, and then use

$$W_{ab} = \textstyle\sum_e U_{ae}\lambda_e (U^\dagger)_{eb} = \sum_e U_{ae}\lambda_e U^*_{be} \tag{28}$$

to simplify the recursion relations. Using that

$$U^\dagger U^* = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{29}$$

one finds, e.g.

$$\tfrac{1}{2} W_{ac}\psi_{ac\cdots} = \textstyle\sum_{\underline{e}=1}^{\Lambda} \mu_{\underline{e}} \tilde{\psi}_{\underline{e},\underline{e}+\Lambda,\cdots} \tag{30}$$

$$W_{aa_1}\psi_{a_2 a} = -\textstyle\sum_{\tilde{a}_1, \tilde{a}_2} U^*_{a_1 \tilde{a}_1} U^*_{a_2 \tilde{a}_2} (\lambda_{\tilde{a}_1} \tilde{\psi}_{\tilde{a}_1 \tilde{a}_2}) \tag{31}$$

and

$$(H\Psi)_{a_1 a_2} = (H)_{a_1 a_2, b_1 b_2}\psi_{b_1 b_2} = U^*_{a_1 \tilde{a}_1} U^*_{a_2 \tilde{a}_2} (\tilde{H})_{\tilde{a}_1 \tilde{a}_2, \tilde{c}_1 \tilde{c}_2} \tilde{\psi}_{\tilde{c}_1 \tilde{c}_2}, \tag{32}$$

with $\tilde{H}$ being unitarily equivalent to $H$,

$$\tilde{H}_{\tilde{a}_1 \tilde{a}_2, \tilde{c}_1 \tilde{c}_2} := U^T_{\tilde{a}_1 e_1} U^T_{\tilde{a}_2 e_2} H U^*_{e_1 \tilde{c}_1} U^*_{e_2 \tilde{c}_2}. \tag{33}$$

The second equation in (22) thus takes a form in which the effective operator on the left hand side becomes

$$
\begin{aligned}
(\tilde{H}_2)_{\tilde{a}_1 \tilde{a}_2, \tilde{c}_1 \tilde{c}_2} = {} & (\tilde{H}_B)_{\tilde{a}_1 \tilde{a}_2, \tilde{c}_1 \tilde{c}_2} + (\lambda_{\tilde{a}_2} - \lambda_{\tilde{a}_1})\delta_{\tilde{a}_1 \tilde{c}_1}\delta_{\tilde{a}_2 \tilde{c}_2} \\
& + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}_{\tilde{a}_1 \tilde{a}_2} \lambda_{\tilde{a}_2} \tilde{H}_B^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}_{\tilde{c}_1 \tilde{c}_2} \lambda_{\tilde{c}_1}.
\end{aligned}
\tag{34}
$$

Note that $(\tilde{H}_B)_{\tilde{a}_1\tilde{a}_2,\tilde{c}_1\tilde{c}_2} = \tilde{T}_{\tilde{a}_1\tilde{a}_2,\tilde{c}_1\tilde{c}_2} + V\delta_{\tilde{a}_1\tilde{c}_1}\delta_{\tilde{a}_2\tilde{c}_2}$ is unitarily equivalent to $(T+V)\delta_{\tilde{a}_1\tilde{c}_1}\delta_{\tilde{a}_2\tilde{c}_2}$ (and it may be advantageous to choose a non-canonical representation of the momentum operators $p_{tA} = p_a$ in $T = p_a p_a$, to simplify $\tilde{T}$). The second term is the analogue of the $\lambda\partial_\lambda$-part of the corresponding $H_0$ in the space-independent fermions approach (see e.g. [1]), while the third term exclusively acts between *particle-hole pairs*, as $\tilde{\theta}_{\underline{c}+\Lambda} = \tilde{\theta}^\dagger_{\underline{c}}$ (this feature, including the particle-hole observation, holds also for the higher $k$ equations (16)).

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## References

[1] J. Hoppe, *On the construction of zero energy states in supersymmetric matrix models I, II, III,* hep-th/9709132, 9709217, 9711033.